Unpacking the Mystery of Mathematical Operations

CAMT 2019
San Antonio
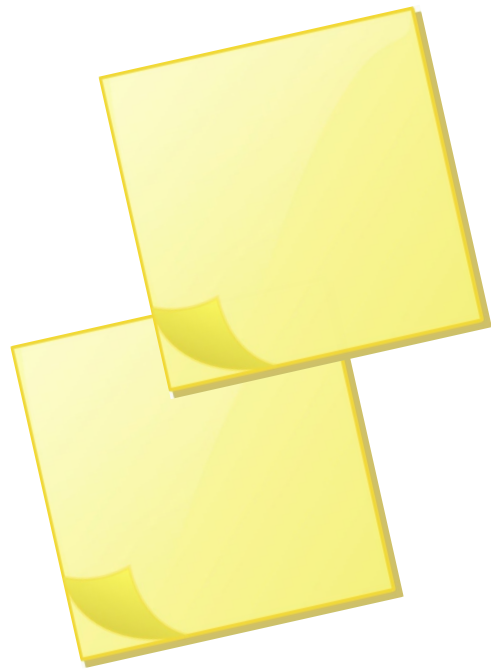
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Take 2 sticky notes and write one multiplication or division word problem on each:

- 1-step problems only

- Use 1- or 2-digit numbers

- Draw a picture to represent your solution

Put them aside for now.

Who's Here Today?

Raise your hand if you are:

- A primary grades teacher
- An intermediate grades teacher
- A middle school teacher
- A high school teacher
- A math coach or specialist
- An administrator
- A teacher educator
- Other??

47% of word problem errors occurred before any sort of calculation process took place.

(Newman, 1977, as cited in Watson, 1980)

Watson, I. (1980). Investigating errors of beginning mathematicians. *Educational Studies in Mathematics*, 11(3), p.319–329. https://doi.org/10.1007/BF00697743

How are these problems the same? Different? Draw a picture and an equation for each.

Mrs. Sandoval handed out 60 stickers at lunch. She gave the first students in line 2 stickers each. How many students got stickers?

Mrs. Sandoval handed out 60 stickers at lunch. She gave the first students in line 30 stickers each. How many students got stickers?

Possible Problem Solutions: Describe the differences

#1	#2	#3	#4	#5	#6	#7	#8	#9	#10	#11	#12	#13	#14	#15	#16	#17	#18	#19	#20	#21	#22	#23	#24	#25	#26	#27	#28	#29	#30
* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *	* *

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Student #1 ******************************

Student #2 ******************************

Mrs. Sandoval handed out 60 stickers at lunch. She gave the first students in line 30 stickers. How many students got stickers?

Multiplier Factor – Measure Factor

Multiplier Factor

- Tells "how many" or "what portion of"
- Like a scale factor
- When a fraction, it acts like an *operator*
- Tells "how many groups"
- Is the divisor in a partitive division problem
- Is the quotient in a measurement division problem
- In Marilyn Burns' "circles and stars" analogy, the multiplier is the "circle"
- In a comparison, it is the "times as many" factor

Measure Factor

- Tells what group you're making a copy of
- When a fraction, is a measurement or a number
- Tells the "size of the group"
- Is the quotient in a partitive division problem
- Is the divisor in a measurement division problem
- In Marilyn Burns' "circles and stars" analogy, the measure is the "stars"
- In a comparison, it is what is being compared

Why does the type of the factor matter?

Why does this model not work?

Make a drawing that shows how you can find the product. Explain your reasoning using word, pictures, and numbers.

3.25 × 5
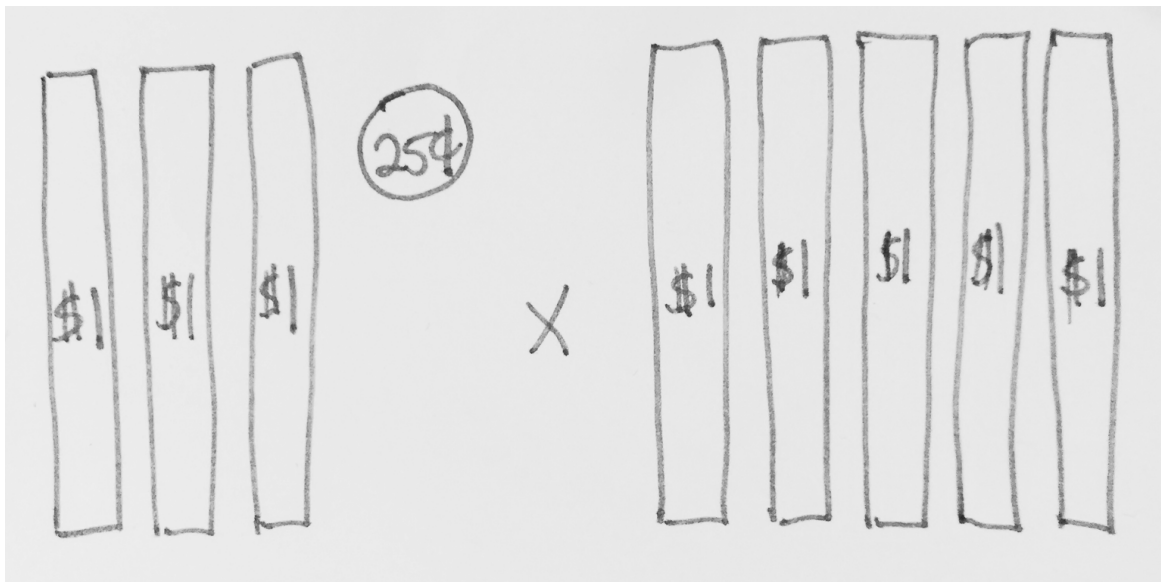
Let's look at the TEKS

Grade 2

(6) Number and operations. The student applies mathematical process standards to connect repeated addition and subtraction to multiplication and division situations that involve equal groupings and shares. The student is expected to:

- (A) model, create, and describe contextual multiplication situations in which equivalent sets of concrete objects are joined; and
- (B) model, create, and describe contextual division situations in which a set of concrete objects is separated into equivalent sets.

Grade 3

(4) Number and operations. The student applies mathematical process standards to develop and use strategies and methods for whole number computations in order to solve problems with efficiency and accuracy. The student is expected to:

- (D) determine the total number of objects when equally-sized groups of objects are combined or arranged in arrays up to 10 by 10;
- (H) determine the number of objects in each group when a set of objects is partitioned into equal shares or a set of objects is shared equally;
- (J) determine a quotient using the relationship between multiplication and division; and
- (K) solve one-step and two-step problems involving multiplication and division within 100 using strategies based on objects; pictorial models, including arrays, area models, and equal groups; properties of operations; or recall of facts.

(5) Algebraic reasoning. The student applies mathematical process standards to analyze and create patterns and relationships. The student is expected to:
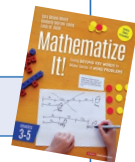
- (B) represent and solve one- and two-step multiplication and division problems within 100 using arrays, strip diagrams, and equations;
- (C) describe a multiplication expression as a comparison such as 3 x 24 represents 3 times as much as 24;
- (D) determine the unknown whole number in a multiplication or division equation relating three whole numbers when the unknown is either a missing factor or product;

Grade 4

(4) Number and operations. The student applies mathematical process standards to develop and use strategies and methods for whole number computations and decimal sums and differences in order to solve problems with efficiency and accuracy. The student is expected to:

- (H) solve with fluency one- and two-step problems involving multiplication and division, including interpreting remainders.

Grade 5

(4) Algebraic reasoning. The student applies mathematical process standards to develop concepts of expressions and equations. The student is expected to:

- (B) represent and solve multi-step problems involving the four operations with whole numbers using equations with a letter standing for the unknown quantity;
- (D) recognize the difference between additive and multiplicative numerical patterns given in a table or graph;

The Work of Multiplication & Division

- *Think about the factors in each situation. What do they represent? What work do they do?*
- *Look at the units in the factors and product in the problem. Do the factors have the same unit or do the factors have two different units?*
  - *If the factors have the same unit, do they create a new unit?*
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FIGURE 1.3 MULTIPLICATION AND DIVISION PROBLEM SITUATIONS

ASYMMETRICAL (NON-MATCHING) FACTORS

	Product Unknown	Multiplier (Number of Groups) Unknown	Measure (Group Size) Unknown
Equal Groups (Ratio/Rate)*	Mayim has 8 vases to decorate the tables at her party. She places 3 flowers in each vase. How many flowers does she need? $8 \times 3 = x$ $x \div 8 = 3$	Mayim has some vases to decorate the tables at her party. She places 3 flowers in each vase. If she uses 24 flowers, how many vases does she have? $x \times 3 = 24$ $x = 24 \div 3$	Mayim places 24 flowers in vases to decorate the tables at her party. If there are 8 vases, how many flowers will be in each vase? $8 \times x = 24$ $24 \div 8 = x$
	Resulting Value Unknown	**Scale Factor (Times as many) Unknown**	**Original Value Unknown**
Multiplicative Comparison	Amelia's dog is 5 times older than Wanda's 3 year-old dog. How old is Amelia's dog? $5 \times 3 = x$ $x \div 5 = 3$	Sydney has $15 to spend at the movies. Her sister has $5. How many times more money does Sydney have than her sister has? $x \times 5 = 15$ $5 = 15 \div x$	Mrs. Smith has 15 puzzles in her classroom. That is 3 times the number of puzzles in Mr. Jackson's room. How many puzzles are in Mr. Jackson's room? $3 \times x = 15$ $15 \div 3 = x$

SYMMETRICAL (MATCHING) FACTORS

	Product Unknown	One Dimension Unknown	Both Dimensions Unknown
Area/Array	Bradley bought a new rug for the hallway in his house. One side measured 5 feet and the other side measured 8 feet. How many square feet does the rug cover? $5 \times 8 = x$ $x \div 8 = 5$	The 40 members of the student council lined up on the stage to take yearbook pictures. The first row started with 8 students and the rest of the rows did the same. How many rows were there? $8 \times x = 40$ $x = 40 \div 8$	Daniella was building a house foundation using her building blocks. She started with 40 blocks. How many blocks long and wide could the foundation be? $x \times y = 40$ $40 \div x = y$
	Sample Space (Total Outcomes) Unknown	**One Factor Unknown**	**Both Factors Unknown**
Combinations** (Fundamental Counting Principle)	Karen has 3 shirts and 7 pairs of pants. How many unique outfits can she make? $3 \times 7 = x$ $3 = x \div 7$	Evelyn says that she can make 21 unique and different ice cream sundaes using just ice cream flavors and toppings. If she has 3 flavors of ice cream, how many kinds of toppings does Evelyn have? $3 \times x = 21$ or $21 \div 3 = x$	Audrey can make 21 different fruit sodas using the machine at the restaurant. How many different flavorings and sodas could there be? $x \times y = 21$ $x = 21 \div y$

Which Element is Unknown?

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The product?
or
One of the factors?

What about remainders?

- We can put aside the remainder and use the whole number part of the result as the answer.
  - *There are 45 flowers. Each bouquet requires 10 flowers. How many full bouquets can be made?*
- We can use the next highest whole-number value.
  - *There are 75 children going on the field trip. Each bus carries 30 children. How many buses are required for the trip?*
- We can include the remainder as a partial quantity, represented as a fraction or decimal.
  - *John can read about 40 pages in an hour. How long will it take him to read a 60-page book?*
- The remainder is the solution.
  - *Mrs. King has 25 books to give to 8 students for summer reading. If each student gets the same number of books, how many will she have left?*

How are these problems the same? Different? Draw a picture and an equation for each.

2/3 of a stick of butter is needed to bake one batch of brownies. How many sticks of butter are needed to make six batches of brownies?

I have 6 cookies. 2/3 of the cookies are oatmeal raisin. How many oatmeal raisin cookies do I have?

Possible Problem Solutions: Describe the differences
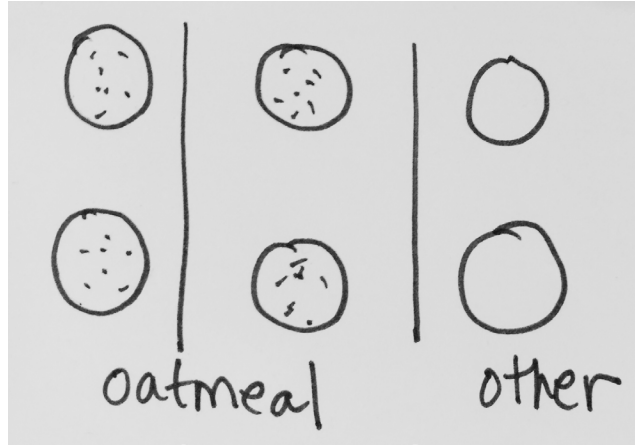
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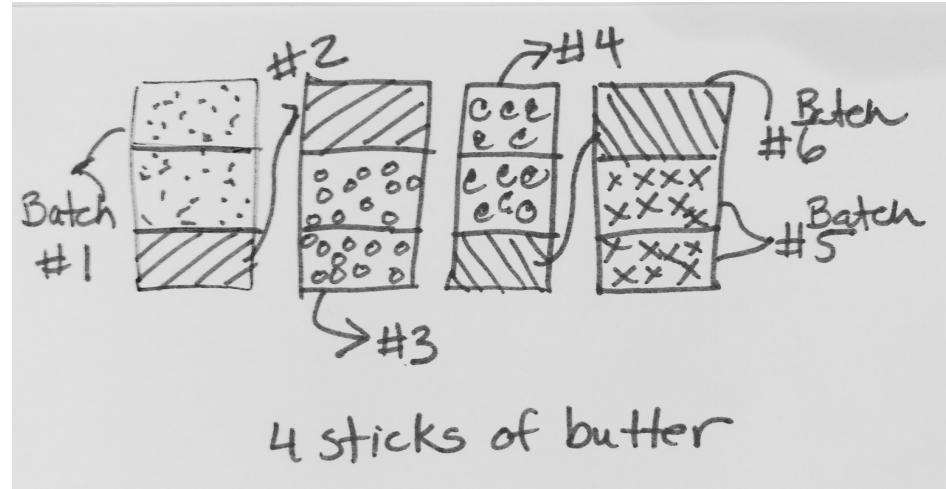
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How are these problems the same? Different? Draw a picture for each.

The 8 children in line each had 6 books to check out from the library. How many books will be checked out?

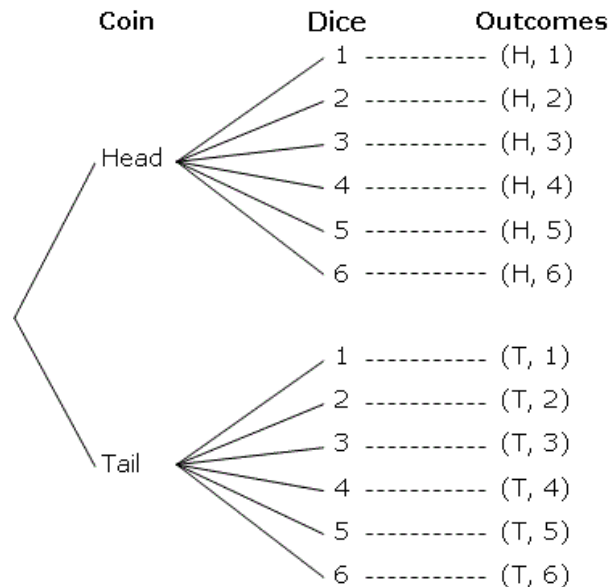The carpet at the front of the classroom measured 6 feet by 8 feet. What is the area of the carpet?

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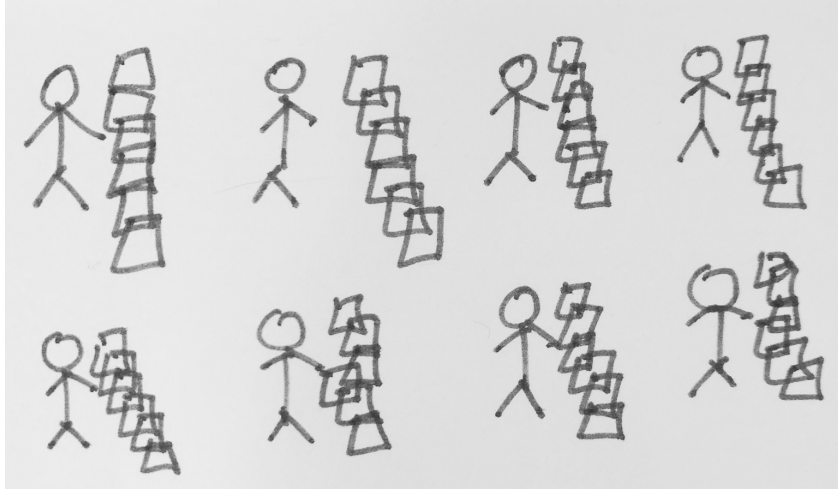
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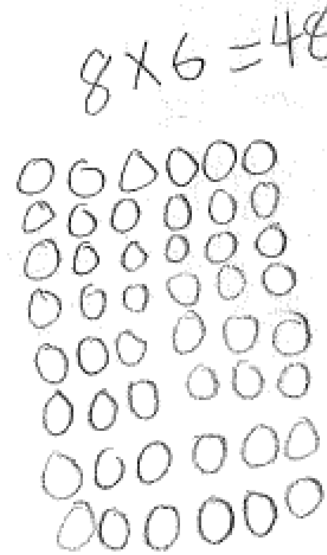
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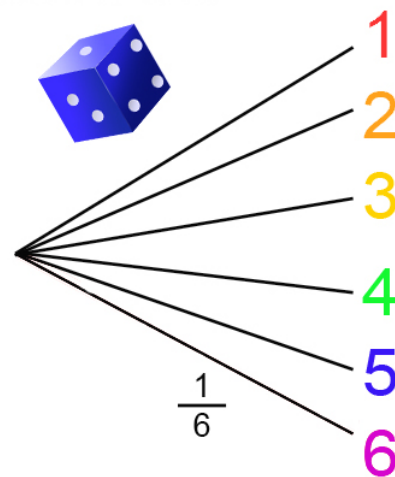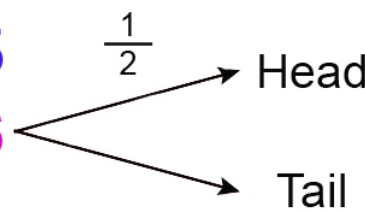|	Product Unknown	One Dimension Unknown		Both Dimensions Unknown
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Combinations

Coin · Dice · Outcomes

Head
1 ----------- (H, 1)
2 ----------- (H, 2)
3 ----------- (H, 3)
4 ----------- (H, 4)
5 ----------- (H, 5)
6 ----------- (H, 6)

Tail
1 ----------- (T, 1)
2 ----------- (T, 2)
3 ----------- (T, 3)
4 ----------- (T, 4)
5 ----------- (T, 5)
6 ----------- (T, 6)

2 × 6 = 12

Throw dice

Toss a coin

1
2
3
4
5
6

$\frac{1}{6}$

$\frac{1}{2}$

Head

Tail

Outcomes
(1, H)
(1, T)
(2, H)
(2, T)
(3, H)
(3, T)
(4, H)
(4, T)
(5, H)
(5, T)
(6, H)
(6, T)

6 × 2 = 12

Big Books

Grade K

Sweet Dreams
Counting
Quantities to Ten

Ten Happy Hens
Subtraction
(Take From)
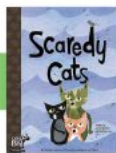
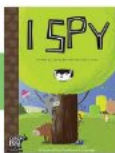*Scaredy Cats*
Combinations
to Ten
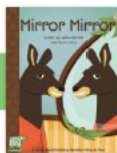
Mirror Mirror
Doubling Numbers
One to Five
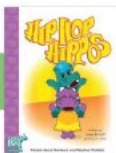
Hip Hop Hippos
Numbers and
Relative Position
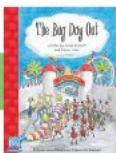
The Bug Day Out
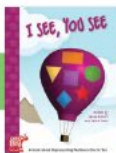Numbers Eleven
to Sixteen

I Spy
Positional
Language

The Clown's New Clothes
Length

Paint a Rainbow
Ordinal Numbers

Mice, Mice Everywhere
Addition
(Put Together)

I See, You See
Identifying
Quantities
One to Ten

Perfect Patterns
Making and
Describing Patterns

Grade 1

Cupcake Capers
Subtraction
(Take From)

Stella's Store
Skip Counting
by Five

A Simple Stew
Growing
Number Patterns

The Best Bug
Non-Standard
Units of Length

The Flower Pot Hen
Representing Data

The Space Party
Multiplication
(Equal Groups)

The Cat Nap
Time on the Hour

Muddy, Muddy Mess
3D Objects and
2D Shapes

Addtron
Using Doubles
to Add

How Many Legs?
Number
Combinations
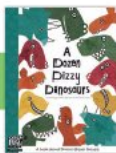
Shoes in Twos
Relating Counting
to Addition

Patterns Here, Patterns There!
Patterns

Grade 2

The Fun Machine
Addition and
Subtraction

A Dozen Dizzy Dinosaurs
Division
(Equal Groups)

Munch and Crunch
Doubling
and Halving

Bears on Buses
Addition
(Add To)

The Pirate's Gold
Division
(Sharing)

Where is Hare?
Position, Direction,
and Movement

Joe's Carrots
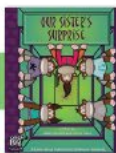Subtraction
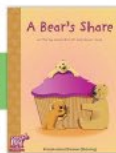(Unknown Addend)

The Big Bug Band
Multiplication
(Arrays)

The Tiny Town Train
Time Past
the Hour

Clowning Around
Collecting and
Representing Data

Our Sister's Surprise
Subtraction
(Unknown Addend)

A Bear's Share
Division
(Sharing)

Revisit Post it Notes

Look again at the problems you put on the post-it notes.
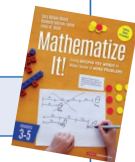
What problem type(s) did you represent?

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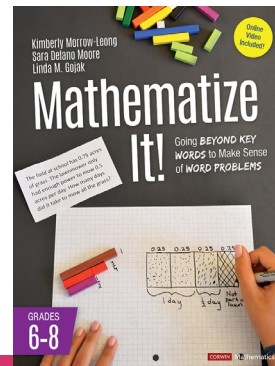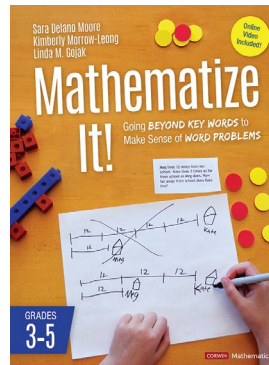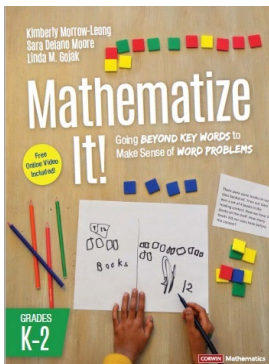|	Product Unknown	One Dimension Unknown	Both Dimensions Unknown
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Thank you!

Slides available in the conference app or at
https://www.origoeducation.com/camt19

Sara Delano Moore, ORIGO Education
(s_moore@origomath.com)
Kimberly Morrow-Leong, George Mason University
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