# Rethinking the Meaning of the Numerator and Denominator

Debi DePaul

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***Rethinking the Meaning of the Numerator and Denominator***



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## Session Goals

Participants will understand that…

- formative assessment should be part of the day-to-day tasks teachers do

Participants will know…

- 3 components of professional judgment
- 5 interpretations for fractions and how the meaning of the numerator and denominator change with each interpretation

Participants will be able to…

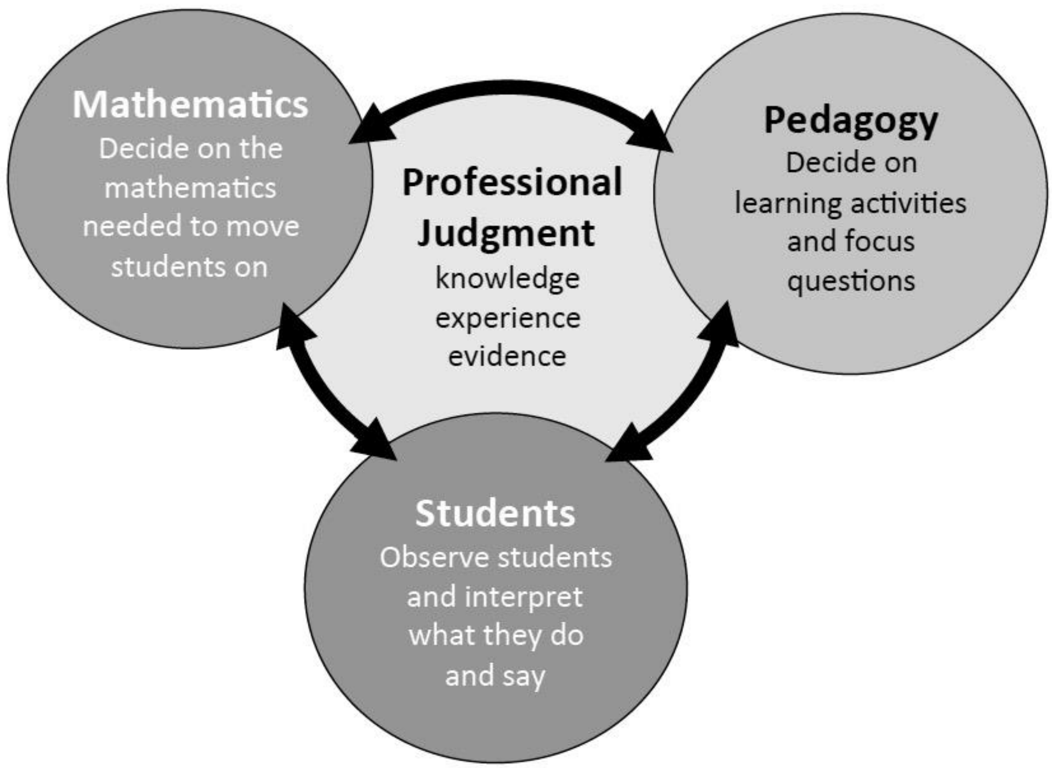
- use the framework of the professional judgment diagram to formatively assess students' fraction understanding

## Professional Learning Should…

- have students' learning as the ultimate goal
- support the ongoing work of teaching
- be grounded in mathematics content
- model and reflect the pedagogy of good instruction
- should create some disequilibrium for teachers
- encourage teacher collaboration
- take into account teachers' contexts
- make use of the knowledge and expertise of teachers
- be sustained and cohesive
- continue over the course of a teacher's career

Smith, Margaret Schwan. *Practice-Based Professional Development for Teachers of Mathematics* NCTM, 2001

# Professional Judgments




**MATHEMATICS**
concepts and practices

- What are the mathematical concepts and practices students need to know and develop?
- What grade levels are they expected to know and develop them?

**STUDENTS**
learning progression
assessing understanding

- What is the typical learning progression students go through to understand the content?
- What are some common misconceptions?
- How can I determine where a student is on the learning progression?

**PEDAGOGY**
developing understanding

- How do I help students deeply understand the concepts and develop fluency with the practices?
- How do I help students avoid or overcome misconceptions?

# Rethinking the Meaning of the Numerator and Denominator

By Debi DePaul

Researchers and classroom teachers do not get many chances to share insights and capitalize on the knowledge gained from each other's work. This article is intended to strengthen the link between the researcher and the practitioner by collecting and organizing the work of published researchers on two big ideas of fraction understanding.

**BIG IDEA #1: Fractions can and must be interpreted in different ways.**

**BIG IDEA #2: The numerator and denominator take on different meanings with the different interpretations.**

## Insights from Researchers

Fractions can have different interpretations depending on the context. Researchers (Kieran, 1998; Lamon, 1999) have identified several of these interpretations. A student's interpretation affects how they think about the numerator and denominator. There are in fact several different interpretations and all should be taught at certain stages of schooling.

## INTERPRETATION 1: Fractions as Part of a Whole

*Fractions as part of a whole* is the first interpretation that many students have of fractions. This understanding is readily accessible because of everyday experiences such as sharing a handful of marbles, folding a piece of paper, cutting a pizza, etc. Each of these scenarios creates parts from an original whole. With this interpretation, the numerator is the number of equal-sized parts, which is indicated in some way such as shading, and the denominator is the number of equal-sized parts in the whole.

Unfortunately, this interpretation is often the only understanding that students have of fractions. It tends to dominate the typical experiences students have when learning fractions. While this is an important understanding, it is not the only one, and a limited repertoire of fraction interpretations puts students at risk for not progressing in higher-level mathematics.

*Fractions as Part of a Whole*
Numerator: Number of equal-sized parts indicated

Denominator: Number of equal-sized parts in the whole

Meaning of $3/4$: 3 parts out of 4 equal parts

## INTERPRETATION 2: Fractions as Measures or Numbers

Figure 1

Interpreting *fractions as measures or numbers* moves the learner to thinking of a fraction as a quantity compared to a referent unit. When a fraction refers to a measure, it compares a certain quantity of an attribute (such as length, capacity, mass, area, or number) to a designated unit (such as a centimeter, cup, gram, square foot, or the quantity. For instance, $3/4$ of a cup of flour means that there are 3 repeats of the referent unit, one-fourths cup, as shown in Figure 1. The numerator indicates the count of unit fractions, whereas the denominator indicates the number of unit fractions needed to create the whole. In this case the referent unit, one-fourths cup, takes 4 repeats to make a whole cup.

*Fractions as Measures or Numbers*
Numerator: Number of repetitions of the unit fraction

Denominator: Number of repetitions needed to create the whole

Meaning of $3/4$: 3 repeats of the unit fraction $3/4$ OR $1/4 + 1/4 + 1/4$

## INTERPRETATION 3: Fractions as Quotients

Debi DePaul

Fractions can also represent the quotient of two quantities. This interpretation opens up the opportunity for fractions to be thought of in two different ways. For instance, the fraction $3/4$ can be understood to be the equivalent of the expression $3 \div 4$ or it can represent the result of doing the division. The numerator is the number of items to be divided or shared and the denominator is the number of equal-sized portions or shares as Figure 2 shows.

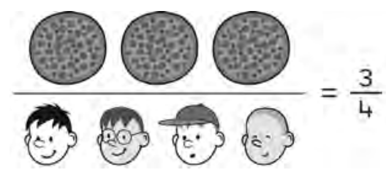



Figure 2

**Problem:** *Suppose that four people are going to share three pizzas. If the pizzas are shared equally, how much pizza does each person get?*

While there are a lot of ways to solve this problem, one of the simplest is to perform the operation $3 \div 4$. The number 3 represents the number of pizzas to be shared and 4 is the number of sharers. It turns out that the answer, $3/4$, is the amount of pizza that each person receives, as shown in Figure 3.

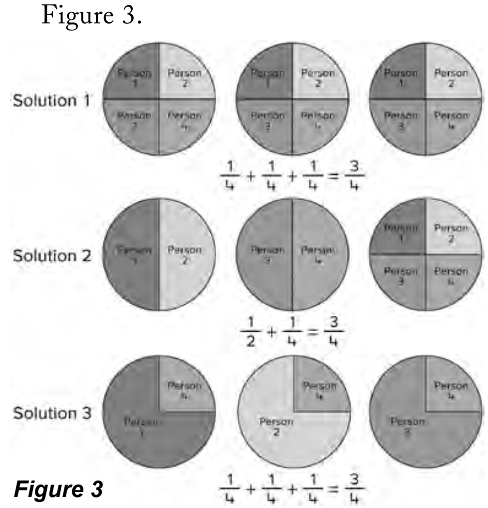



Figure 3

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## *Rethinking the Meaning of the Numerator and Denominator*

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A student who believes that a number cannot be divided by a greater number may resist the notion of dividing 3 by 4 to work out the answer. Interpreting *fractions as quotients* requires that students understand that it is okay to do so, this is often contrary to what they are taught in elementary schools.

*Fractions as Quotients*

Numerator: Number of items to be divided or shared

Denominator: Number of equal-sized partitions or shares

Meaning of $3/4$: $3 \div 4$ OR the result when divided or shared

### INTERPRETATION 4: Fractions as Ratios

Ratios are not numbers at all, but are *relationships* between numbers, which are sometimes written in fraction form. When fractions represent ratio relationships, the rules we use for operating on typical fractions often do not apply to *fractions as ratios*. For instance, suppose that in one classroom there is a ratio of three boys for every four girls and in the neighboring classroom there is a ratio of two boys to three girls, as shown in Figure 4. These ratios can be written in the form $3/4$ and $2/3$, respectively.

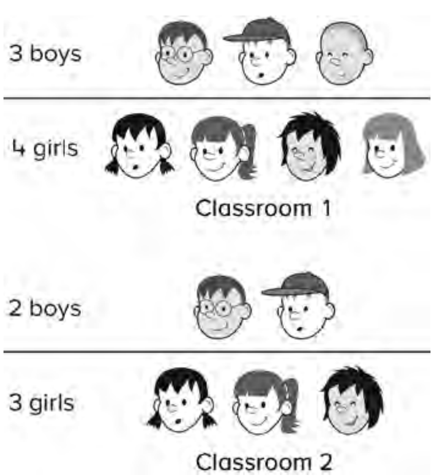


**Figure 4**

Adding the fractions in the typical way to find the ratio of the boys to the girls in both classrooms together would not make sense for this interpretation as the process results in having 17 boys for every 12 girls.

$$\frac{3}{4}+\frac{2}{3}=\frac{9}{12}+\frac{8}{12}=\frac{17}{12}$$

**Figure 5**

Figure 6 illustrates that the correct ratio of boys to girls in both classrooms combined is $5/7$.

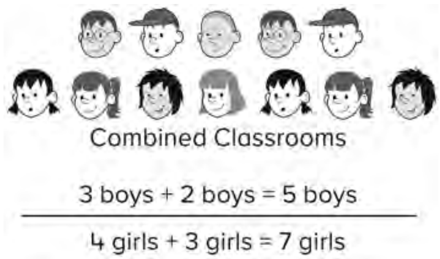


**Figure 6**

Using fractions to indicate a ratio relationship can be confusing to some students. They have to understand the context to make sense of what the numbers in the fraction mean since both the numerator and the denominator represent either a part or a whole. It can be helpful to label what the quantities are that are getting compared when writing fractions that represent ratio relationships, as seen above.

*Fractions as Ratios*

Numerator: Some related quantity (part or whole)

Denominator: Some related quantity (part or whole)

Meaning of $3/4$: 3 quantities for every 4 quantities

### INTERPRETATION 5: Fractions as Operators

One interpretation of fractions is that of an *operator*, or a set of instructions for carrying out a process. The notion of an operator is about shrinking and enlarging, or multiplying and dividing. For example, $3/4$ *of* can be interpreted as the process of multiplying by 3 and dividing the result by 4 or the equivalent process of dividing by 4 and then multiplying the result by 3. Notice that the result is the same either way (Figures 7 and 8). Knowing that there are different, but equivalent, processes that yield the same result can be helpful when computing with fractions, but students need to understand why they work.

*Fractions as Operators*

Numerator: Factor of increase

Denominator: Factor of decrease

Meaning of $3/4$: increase by a factor of 3, then decrease by a factor of 4 OR decrease by a factor of 4, then increase by a factor of 3 OR increase by a factor of $3/4$ (which results in a decrease)

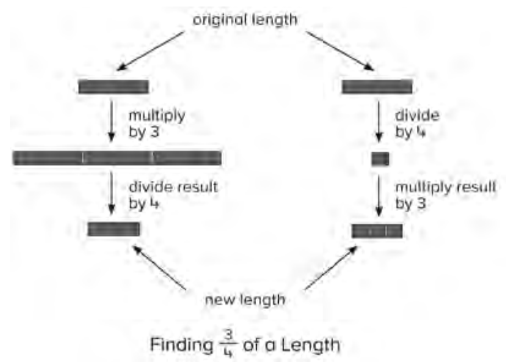


**Figure 7**

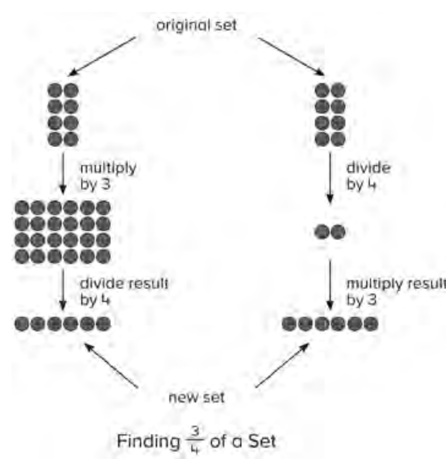


**Figure 8**

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## Fraction Interpretations

| Fractions as… | Numerator | Denominator | Meaning of 3/4 |
|---|---|---|---|
| **Part of a Whole** | number of equal-sized parts indicated | number of equal-sized parts in the whole | 3 parts out of 4 equal-sized parts |
| **Numbers or Measures** | number or count of equal-sized parts (unit fractions) | number of equal-sized parts (unit fractions) needed to create the whole | • 3 counts (repetitions) of the unit fraction 1/4<br>• 1/4 + 1/4 + 1/4 |
| **Quotients** | number of items in the whole | number of shares or equal-sized parts | • 3 ÷ 4<br>• the result when divided or shared |
| **Ratios** | some related quantity (part or whole) | some related quantity (part or whole) | 3 quantities for every 4 quantities |
| **Operators** | factor of increase | factor of decrease | • increase by a factor of 3, then decrease by a factor of 4<br>• decrease by a factor of 4, then increase by a factor of 3<br>• increase by a factor of 3/4 |

# Next Steps

| Student Name | What do you notice?<br><br>How does the student think about the numerator and denominator? | What follow up questions do you have?<br><br>What follow up task will I have the student do next? | How am I going to help this student move forward? |
|---|---|---|---|
| | | | |
| | | | |
| | | | |

# **Task 2** Zoo Animals

8 years old

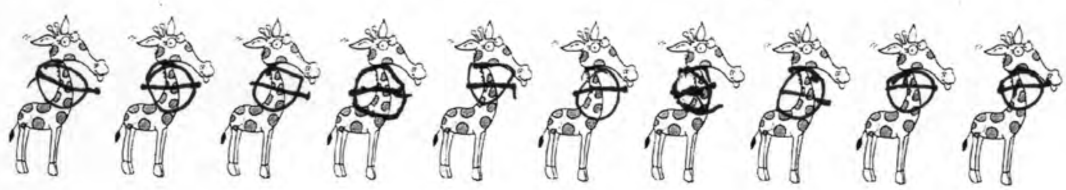
NAME: ~~8~~ Eddie DATE: 7-7-2010

Put a circle around half of the zebras.

What makes this a half?

~~8~~ zerbras bodies are ~~hafe~~ halfs.

Put a circle around half of the giraffes.



What makes this a half?

a giraffes neck are Halfs.

*Half of the zebras* looks different from *half of the giraffes*.
How can they both be half?

Treacy, Kaye. Revealing what Students Think
Diagnostic Tasks For Fractional Numbers, 2009

# Task 7 Which Is Bigger?

NAME: Yesenia 5th grade DATE: 6-9-09

Tanay and Brit had been working on fractions at school. After school they were chatting about what they had learned.


**Tanay** **Brit**

Who is right? Brit

like if you had a pizza?! and what would you want more or less?

Explain to Tanay and Brit how you know which is bigger. You might like to draw a diagram to help.

| Tanay | Brit |
|---|---|
| $\frac{1}{3}$ has less peice | more peices |

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3rd

# Task 8 Cooking at Home

NAME: miguel DATE: 6-2-09

**Part One**

You are cooking at home with your Nonna. A cookie recipe uses $\frac{1}{3}$ cup of nuts and $\frac{1}{4}$ cup of raisins.

Which is the bigger: the amount of nuts or the amount of raisins?

Draw a picture to explain which is bigger.

$\frac{1}{3}$ $\frac{1}{4}$

demonstration

**Part Two**

A cake recipe uses $\frac{3}{4}$ cup of chopped apple and $\frac{2}{3}$ cup of milk.

Which is bigger: the amount of milk or the amount of chopped apple?

Draw a picture to explain which is bigger.

"The bottom number tells us how many to draw. The top number tells us how many to cross off. They are the same because there is one left."

$\frac{3}{4}$ Same $\frac{2}{3}$

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# Task 8 Cooking at Home

10th

NAME: Everica DATE:

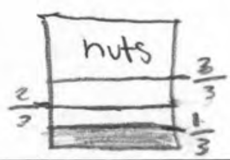
**Part One**

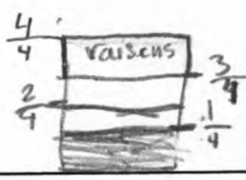
You are cooking at home with your Nonna. A cookie recipe uses $\frac{1}{3}$ cup of nuts and $\frac{1}{4}$ cup of raisins.

Which is the bigger: the amount of nuts or the amount of raisins?

Draw a picture to explain which is bigger.

$\boxed{\text{raisins}}$ is more because $\frac{1}{4}$ is bigger $\frac{1}{3} < \frac{1}{4}$




**Part Two**

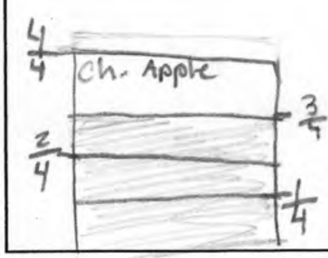
A cake recipe uses $\frac{3}{4}$ cup of chopped apple and $\frac{2}{3}$ cup of milk.

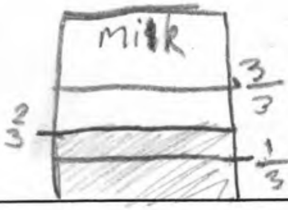
Which is bigger: the amount of milk or the amount of chopped apple?

Draw a picture to explain which is bigger.

$\frac{3}{4} > \frac{2}{3}$

the numerator and denomenator for chopped apple is bigger than $\frac{2}{3}$.



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# Task 8 Cooking at Home

10th

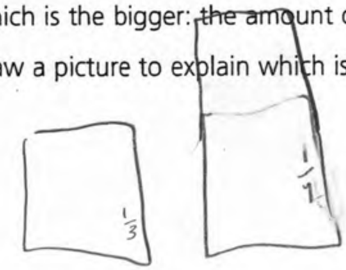
NAME: Luis DATE:

**Part One**

You are cooking at home with your Nonna. A cookie recipe uses $\frac{1}{3}$ cup of nuts and $\frac{1}{4}$ cup of raisins.

Which is the bigger: the amount of nuts or the amount of raisins?

Draw a picture to explain which is bigger. 4 is a bigger denometor




**Part Two**

A cake recipe uses $\frac{3}{4}$ cup of chopped apple and $\frac{2}{3}$ cup of milk.

Which is bigger: the amount of milk or the amount of chopped apple?

Draw a picture to explain which is bigger. 3/4 becuase 3 is bigger than 2, 4 is bigger than 3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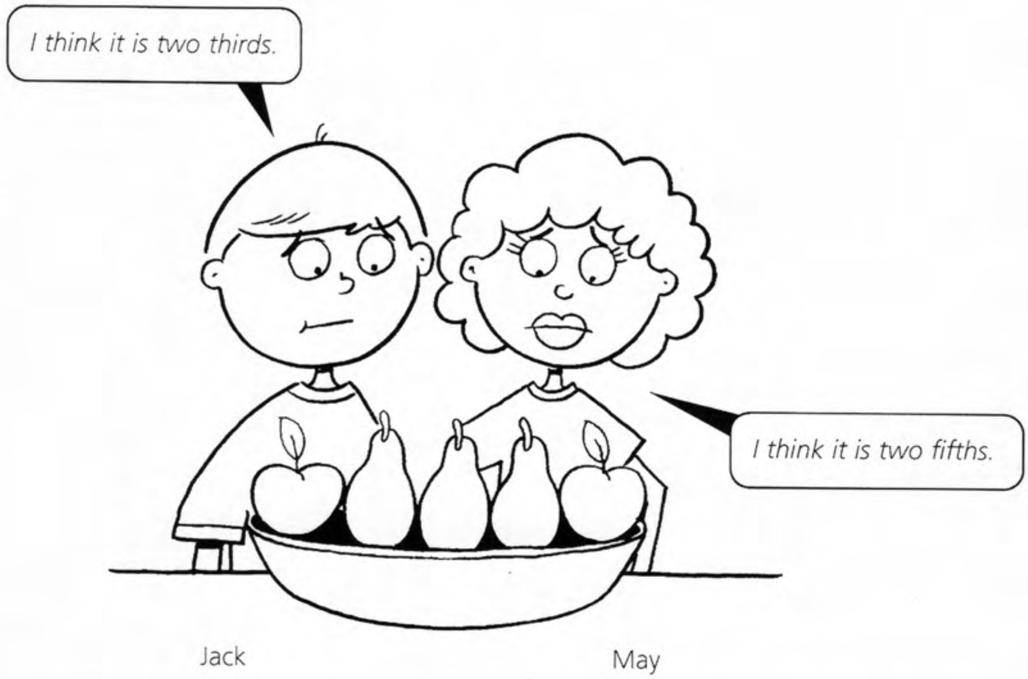
# Task 9 Fruit Bowl

NAME: Morgan 12 yrs old DATE: 8-4-10

What fraction of the fruit in the bowl is apples?




Who is right and why? May is Right becaus in all there are 5 Bruits in the bowl and 2 of them are apples

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Grade Level:
9 10 11 12

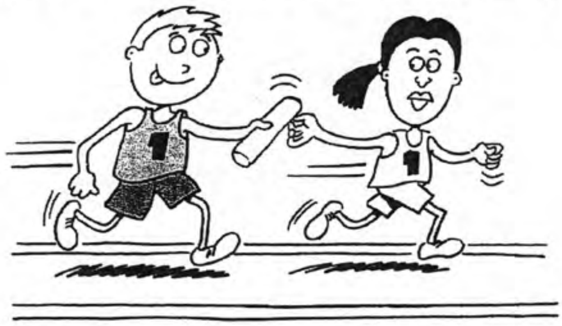
# Task 14 Running a Race

| NAME: Gladys | DATE: 6/4/09 |
|---|---|

**Part One**

A team of runners is running a race that is $1\frac{1}{2}$ kilometres long.

Each runner runs $\frac{1}{8}$ of a kilometre. How many runners are needed for the team?



Draw a diagram below to work it out or to explain your thinking.

12 because

12/8 = 1 1/2

**Part Two**

If there were 8 runners in a team, and each person ran $\frac{1}{3}$ of a kilometre, how long would the race be? Draw a diagram below to work it out or to explain your thinking.

1/3 1/3 1/3 1/3 1/3 1/3 1/3 1/3

8/3 2 2/3

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# Task 14 Running a Race

NAME: Brynne DATE:

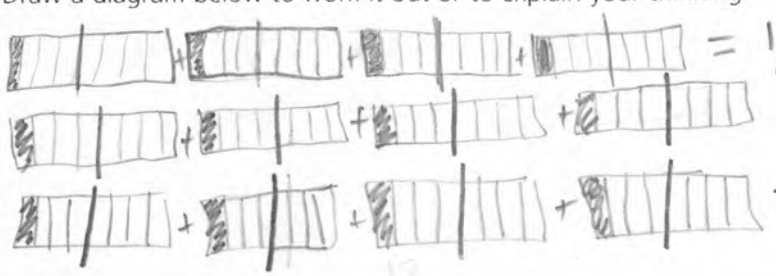
**Part One**

A team of runners is running a race that is $1\frac{1}{2}$ kilometres long.

Each runner runs $\frac{1}{8}$ of a kilometre. How many runners are needed for the team?

Draw a diagram below to work it out or to explain your thinking.

12 every box is a person.



$= 1/2$

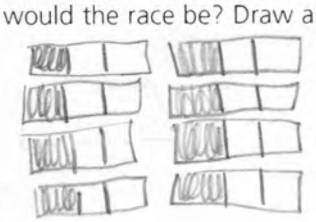
$1 + 1/2 = 1\frac{1}{2}$

$= 1$

12

**Part Two**

If there were 8 runners in a team, and each person ran $\frac{1}{3}$ of a kilometre, how long would the race be? Draw a diagram below to work it out or to explain your thinking.



8/24

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Grade Level:
9 10 11 12

# Task 14 Running a Race

NAME: JJ DATE:

**Part One**

A team of runners is running a race that is $1\frac{1}{2}$ kilometres long.

Each runner runs $\frac{1}{8}$ of a kilometre. How many runners are needed for the team?

Draw a diagram below to work it out or to explain your thinking.

$3/2 \div 1/8 = 12/8$

12 Runners

**Part Two**

If there were 8 runners in a team, and each person ran $\frac{1}{3}$ of a kilometre, how long would the race be? Draw a diagram below to work it out or to explain your thinking.

$8 \times 1/3 = 2\frac{2}{3}$

$2\frac{2}{3}$ Kilometers.

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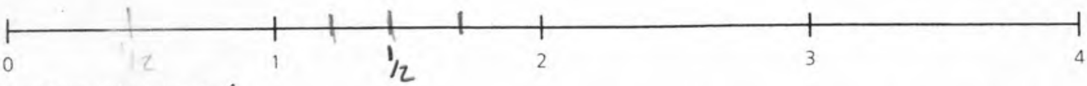
# Task 22 Number Lines

NAME: Evelyn DATE: 6-5-09

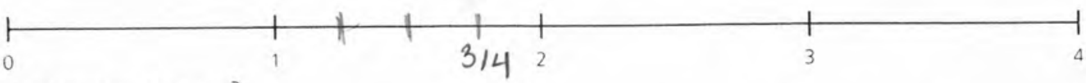
Show the number $\frac{1}{2}$ on the number line below.




Explain why this is $\frac{1}{2}$.

Where I put the number half is half because if I divide the unit in 4 half would be 1/2 on 4/4 so it would be like 2/4 but in a diffrent way of writting it.

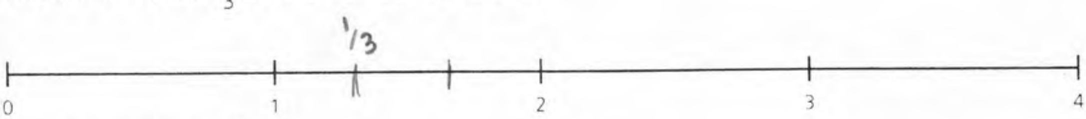
Show the number $\frac{3}{4}$ on the number line below.




Explain why this is $\frac{3}{4}$.

I put 3/4 here becaus if I divide the unit in 4 3/4 would be the third one. For example 1/4, 2/4, 3/4, 4/4

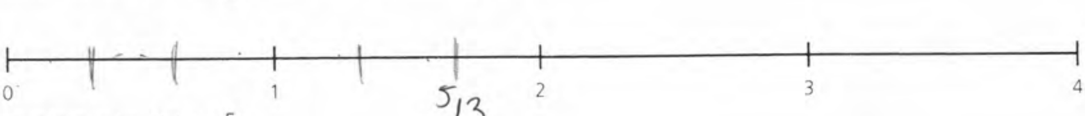
Show the number $\frac{1}{3}$ on the number line below.




Explain why this is $\frac{1}{3}$.

I think this is one third because I divided the unit in to 3 and 1/3 would be 1 of the unit.

Show the number $\frac{5}{3}$ on the number line below.




Explain why this is $\frac{5}{3}$.

I think this is 5/3 because

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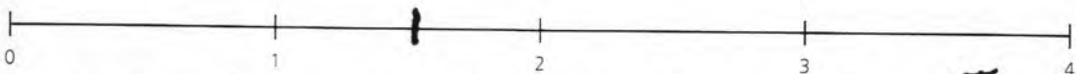
# Task 22 Number Lines

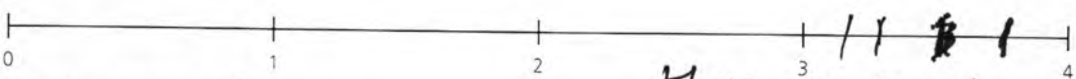
NAME: Suzy 9 years old DATE: 7-7-10

Show the number $\frac{1}{2}$ on the number line below.



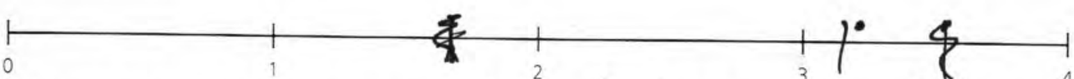

Explain why this is $\frac{1}{2}$.
on the number one is like one whole then I made a line in the middle to show 1 half

Show the number $\frac{3}{4}$ on the number line below.



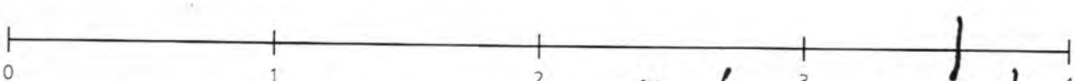

Explain why this is $\frac{3}{4}$.
There is 3 whole pieces then there is a half piece in the middle of 3 and 4

Show the number $\frac{1}{3}$ on the number line below.




Explain why this is $\frac{1}{3}$. because it is $\frac{1}{3}$ because it's 3 whole pieces and 1 piece left

Show the number $\frac{5}{3}$ on the number line below.




Explain why this is $\frac{5}{3}$. because it is $\frac{5}{3}$ because it's has 3wholes like 3 whole pies and 5pieces left



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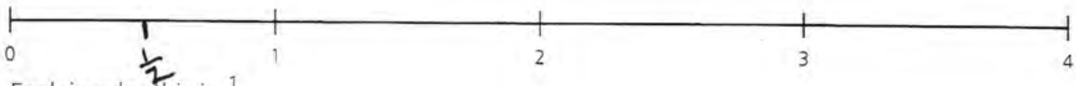
# Task 22 Number Lines

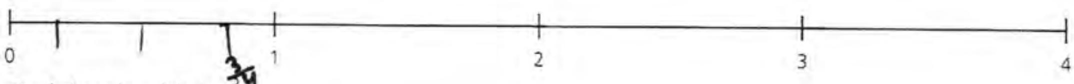
NAME: Morgan 12 DATE: 8-4-10

Show the number $\frac{1}{2}$ on the number line below.



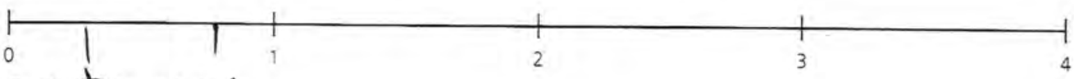

Explain why this is $\frac{1}{2}$.

Show the number $\frac{3}{4}$ on the number line below.



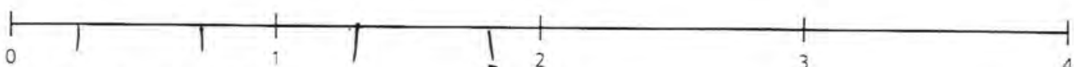

Explain why this is $\frac{3}{4}$.

Show the number $\frac{1}{3}$ on the number line below.




Explain why this is $\frac{1}{3}$.

Show the number $\frac{5}{3}$ on the number line below.




Explain why this is $\frac{5}{3}$.



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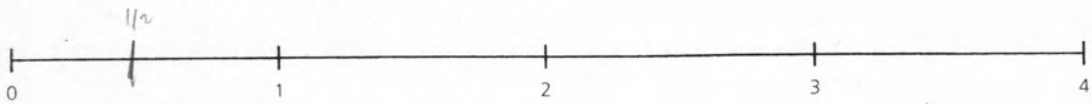
# Task 22 Number Lines

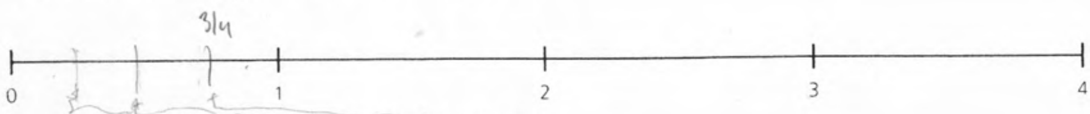
NAME: Vivian DATE:

Show the number $\frac{1}{2}$ on the number line below.



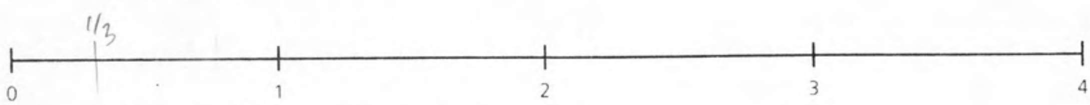

Explain why this is $\frac{1}{2}$. It is half of the number.

Show the number $\frac{3}{4}$ on the number line below.



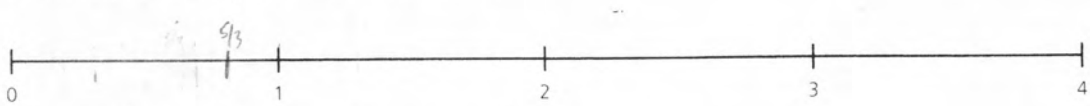

Explain why this is $\frac{3}{4}$. Out of 4 spaces, the 3/4 would be the 3rd line.

Show the number $\frac{1}{3}$ on the number line below.




Explain why this is $\frac{1}{3}$. It is out of 3 spaces it's the 1st line.

Show the number $\frac{5}{3}$ on the number line below.




Explain why this is $\frac{5}{3}$. It's a little more than 3/4.

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# Task 25 Party Food

NAME: Nelson DATE: 6509

There were three slices of garlic bread at a party and four children who wanted some. Share out the garlic bread so that everyone gets an even share.

How much garlic bread does each person get?

3/4

Explain how you worked it out.

I cut all bread into 4 pices cause theres 4 peopleand each got a 1/4 Slice from each bread and theres only 3 breads its 3/4

There were two pizzas at the party. Jesse said that he did not like pizza. Share out the pizza between the three other children.

How much pizza does each person get?

2/3

Explain how you worked it out.

Cut the pizza 3 pices cause theres 3 people each got 1/3 slice from each pizza and theres only 2 pizzas its 2/3

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# Task 25 Party Food

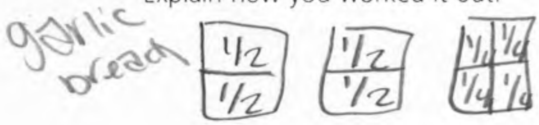
NAME: Jack DATE: 6/5/09

There were three slices of garlic bread at a party and four children who wanted some. Share out the garlic bread so that everyone gets an even share.

How much garlic bread does each person get?

1/2 and 1/4

Explain how you worked it out.




There were two pizzas at the party. Jesse said that he did not like pizza. Share out the pizza between the three other children.

How much pizza does each person get?

2/3

Explain how you worked it out.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# Task 28 Visit to the Zoo

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NAME: Maria DATE: 6-3-09

To take a class to visit the zoo, we have to have one adult for every six students.


What fraction of the group would be adults?

Explain how you know.

1/7

6kids and 1 adult

What fraction would be students?

Explain how you know.

6/7

6 kids and 1 adult

What is the ratio of adults to students? 1:6

If we had three classes going together, adding up to 96 students, how many adults would be needed?

96 ÷ 6 = 16 adults

What fraction of this group would be adults?

Explain how you know. 16/96

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# Task 29 Making Lemonade

NAME: Dylan DATE: 6-4-09

Some children were making lemonade for the school fundraiser. They made 1 litre of lemonade using $\frac{1}{5}$ lemon juice to $\frac{4}{5}$ water and thought that this tasted just right. However, they need to make 10 litres for the fundraiser.


How much lemon juice and how much water would they need to use to make it taste the same?

Lemon juice $\frac{10}{50}$ Water $\frac{40}{50}$

Use the space below to show how you worked it out.

$\frac{1 \cdot 10}{5 \cdot 10} = \frac{10}{50}$ lemon juice

$\frac{4 \cdot 10}{5 \cdot 10} = \frac{40}{50}$ water

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Diagnostic Tasks For Fractional Numbers, 2009

# Task 29 Making Lemonade

NAME: CherDen DATE: June 4

Some children were making lemonade for the school fundraiser. They made 1 litre of lemonade using $\frac{1}{5}$ lemon juice to $\frac{4}{5}$ water and thought that this tasted just right. However, they need to make 10 litres for the fundraiser.


How much lemon juice and how much water would they need to use to make it taste the same?

Lemon juice 2 Water 8

Use the space below to show how you worked it out.

$\frac{1}{5} \times \frac{10}{1} = \frac{10}{5}$ 2 Lemon Juice

$\frac{4}{5} \times \frac{10}{1} = \frac{40}{5}$ 8 water

\\\\\\\\\\

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